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Running couplings in Higher derivatives theories

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D.B., J. Donoghue, G. Menezes, R. Percacci, *Physical running of couplings in quadratic gravity*, Phys.Rev.Lett. 133 (2024) 2, 021604, arXiv:2403.02397

D. B., J. F. Donoghue, G. Menezes and R. Percacci, *Renormalization and running in the 2D CP(1) model*, *JHEP* 02 (2025) 146, arXiv: 2408.13142

D. B., L. Parente and O. Zanusso, *Physical Running in Conformal Gravity and Higher Derivative Scalars*, *Phys.Rev.D* 111 (2025), arXiv: 2410.21475

Why higher derivative?

- Einstein general relativity as a QFT is perturbatively non renormalizable

- Higher derivative operators R^2, $R_{\mu\nu}R^{\mu\nu}$ and $R_{\mu\nu\rho\lambda}R^{\mu\nu\rho\lambda}$ contain a fourth derivative kinetic term for the metric

$$S = \int d^4x\sqrt{-g}\left[+2\Lambda - Z_N R + \frac{1}{2\lambda}C^2 + \frac{1}{\xi}R^2 - \frac{1}{\rho}E\right]$$

$$\Downarrow$$

The theory is now perturbatively renormalizable [Stelle, ‘77]

- Ostrogradsky instability, ghosts and breakdown of unitarity

Renormalization: historical overview

RG equations of quadratic gravity:

- Julve & Tonin, ’78: perturbatively Asymptotically Free (AF) only with scalar tachyon

No Nakanishi-Lautrup ghost

- Fradkin & Tseytlin, ’82: AF without scalar tachyon

Numerical error

- Avramidi & Barvinski, ’85: again AF only with scalar tachyon

FRG results

AF + interacting fixed points [Codello, Percacci, ‘05] [Groh, Rechenberger, Saueressig, Zanusso, ‘11]

	λ_*	ξ_*	ρ_*	ω_*	$\tilde{Z}_{N*}$	$\tilde{G}_*$
FP_1	0	0	0	-0.02286	0.00833	2.388
FP_2	29.26	-220.2	0	0.4040	0.01318	1.509
FP_3	52.61	1672	0	-0.0944	0.00761	2.614

FP_1	4	2	0	0	0
FP_2	$2.352 + 1.677i$	$2.352 - 1.677i$	1.767	0	-3.200
FP_3	$2.327 + 1.521i$	$2.327 - 1.521i$	1.237	0	-5.277

[K. Falls , N. Ohta, R.Percacci, arXiv:2004.04126 [hep-th]]

Problem:

Do these RG equations reproduce the momentum dependence of scattering amplitudes?

Running couplings from amplitudes (Gell-Mann & Low)

Consider a scattering amplitude renormalized with a momentum subtraction scheme at a mass scale m

$$\mathrm{A}(g, m, p) \sim g + \boxed{g^2 \log\left(\frac{p^2}{m^2}\right)}$$

if $p \gg m$
Breakdown of perturbativity

Large logs are reabsorbed in $g(\bar{p}) = g_i + g_i^2 \log(\bar{p}^2/m_j^2)$ at a new energy scale $\bar{p} \sim p$

$$\mathrm{A}(g, m, p, \bar{p}) \sim g(\bar{p}) + g^2(\bar{p}) \log\left(\frac{p^2}{\bar{p}^2}\right)$$

The momentum running of g is defined by integrating an infinitesimal shift of $\bar{p}$:

$$\bar{p}\frac{d}{d\bar{p}} g(\bar{p}) = \beta^{\bar{p}}(g) \qquad g(\bar{p}') = \frac{g(\bar{p})}{1 - \beta^{\bar{p}}(g) \log\left(\frac{\bar{p}'^2}{\bar{p}^2}\right)}$$

Extended perturbative regime

Running of coupling from regulators (Wilson)

Regularization and renormalization introduce an unphysical energy scale (Λ, k, μ,…)

A typical renormalized amplitude in $\overline{MS}$ is

$$A(g_i, \mu, p, m) \sim g^2(\mu) + g^2(\mu)\, c \log\left(\frac{f(m,p)}{\mu^2}\right)$$

Physics independent of μ ⇒ Callan-Symanzik equations

$$\mu \frac{d}{d\mu} A(g_i, \mu, p) = 0, \qquad \beta^{\mu}(g) \coloneqq \mu \frac{\partial}{\partial \mu} g(\mu)$$

Nonperturbative version: Functional Renormalization Group (FRG)

$$S[\phi] \to S[\phi] + \phi R_k \phi \qquad k \frac{d\Gamma_k}{dk} = \frac{1}{2} Tr\left(\frac{\delta^2 \Gamma_k}{\delta\phi\delta\phi} + R_k\right)^{-1} k \frac{dR_k}{dk}$$

Universality of running couplings

In massless theories only one dimensionless quantity p^2/μ^2

$$\beta^{\bar{p}}(g) = \beta^{\mu}_{m=0}(g)$$

In the presence of other energy scales:

With $\overline{MS}$ prescription

$$A(g,\mu,p,m) \sim g(\mu) + g^2(\mu)\, b \log\left(\frac{m^2}{\mu^2}\right) + g^2(\mu) c \log\left(\frac{p^2}{m^2}\right)$$

$$A(g,\mu,p,m)\Big|_{\mu=p} = g(p) + g^2(b-c)\boxed{\log\left(\frac{m^2}{p^2}\right)}$$

In momentum subtraction schemes (p-running)

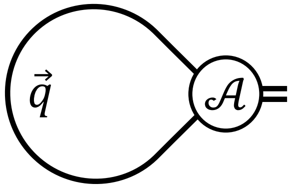
$$g(m) \coloneqq A(g,\mu,p,m)\Big|_{p=m} = g(\mu) + g(\mu) b \log\left(\frac{m^2}{\mu^2}\right)$$

$$A(g,p,m) = g(m) + g^2(m) c \log\left(\frac{p^2}{m^2}\right) \coloneqq g_i(p)$$

In 2-derivative theories in $d=4$, $b=c$ if $p \gg m$ $\Rightarrow$ universality

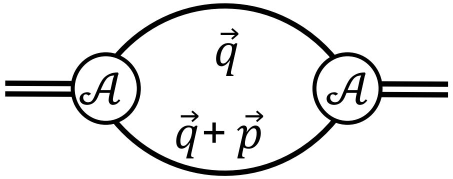
Other cases
(ex. higher derivative)?

Higher derivatives theories



$$\mu^{2\epsilon}\int d^{4-2\epsilon}q\,\frac{\mathcal{A}}{m^2q^2+q^4}$$
$$\sim\frac{\mathcal{A}}{\epsilon}+\mathcal{A}\log\left(\frac{\mu^2}{m^2}\right)$$

No running with p

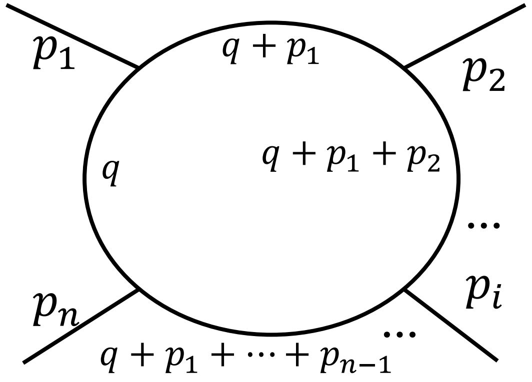


$$\mu^{2\epsilon}\int d^{4-2\epsilon}q\,\frac{\mathcal{A}^2}{(m^2q^2+q^4)(m^2(q+p)^2+(q+p)^4)}$$
$$\sim\frac{\mathcal{A}^2}{p^4}\log\left(\frac{m^2}{p^2}\right)+O\left(\frac{m}{p}\right)$$

p-running without $\frac{1}{\epsilon}$ poles if $\mathcal{A}\sim p^4$

General diagram

In higher derivatives theories there are off-shell IR divergences:



$$\sim \int d^4q \, \frac{N(p,q_i)}{q^4(q+p_1)^4 \times \cdots \times (q+p_1+\cdots+p_{n-1})^4}$$

$q \ll p_i$ $\qquad\qquad$ $q \gg p_i$

$$\frac{1}{p_1^4 \times \cdots \times (p_1+\cdots+p_{n-1})^4} \int d^4q \boxed{\frac{N(p,q_i)}{q^4}} + \cdots \qquad\qquad \int d^4q \boxed{\frac{N(p,q_i)}{q^{4n}} + \cdots}$$

If IR regulator $\neq$ UV regulator $\qquad$ (ex. $m_i \neq 0$, IR cutoff λ)

$\beta^{\bar{p}}(g) \neq \beta^{\mu}(g)$ $\qquad$ even when $\qquad$ $\bar{p} \gg m_i$

[D.B., John Donoghue, Roberto Percacci, '23]
[John Donoghue, Gabriel Menezes, '23]

What about curved spacetime?

HD scalar in curved spacetime

[D.B., Luca Parente, Omar Zanusso, ’24]

$$S_{HDS} = \frac{1}{2}\int d^4x\sqrt{\bar{g}}\left[\phi\bar{\Box}^2\phi + \phi\bar{\nabla}_\mu\left((\xi_1\bar{R}^{\mu\nu} + \xi_2\bar{g}^{\mu\nu}\bar{R})\bar{\nabla}_\nu\phi\right) + \phi U\phi\right]$$

$$U = \lambda_1\bar{C}^2 + \lambda_2\bar{R}_{\mu\nu}\bar{R}^{\mu\nu} + \lambda_3\bar{R}^2 + \lambda_4\bar{\Box}\bar{R}$$

$$\Gamma_{1-loop}[\bar{g},\varphi] = \frac{1}{2}Tr\log\left(\frac{\delta^2 S}{\delta^2\phi}\right)$$

$$\Gamma_{1-loop}[\bar{g},0] = \int d^4x\sqrt{\bar{g}}\left[\bar{C}_{\mu\nu\rho\sigma}f_\lambda\left(\bar{\Box};\mu^2,m^2\right)\bar{C}^{\mu\nu\rho\sigma} + \bar{R}\, f_\xi\left(\bar{\Box};\mu^2,m^2\right)\bar{R} + O\left(R^3\right)\right]$$

when $\bar{\Box} \gg m^2$,

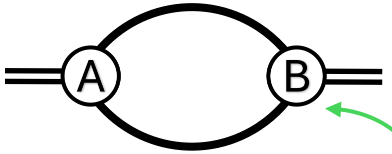
$$f_i(\Box,\mu,m) \sim b_i\log\left(\frac{m^2}{\mu^2}\right) + c_i\log\left(\frac{\Box}{m^2}\right)$$

$$\beta_i^\mu \propto b_i, \quad \beta_i^{\bar{p}} \propto c_i \qquad\qquad b_i = c_i?$$

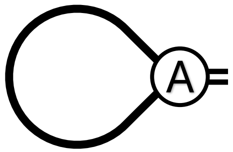
Heat kernel computation

$$\frac{\delta^2 S}{\delta^2\phi} = H = \Box^2 + V^{\rho\lambda}\nabla_\rho\nabla_\lambda + N^\mu\nabla_\mu + U$$

- Heat kernel (HK) technique permits to do manifestly covariant computations in perturbation theory. Divergencies are



$$-\frac{1}{\epsilon}\frac{1}{2(4\pi)^2}\int d^4x\, tr\left[\boxed{\frac{R^2_{\mu\nu\rho\sigma}}{90}-\frac{R^2_{\mu\nu}}{90}+\frac{5R^2}{180}} - \boxed{U} - \boxed{\frac{1}{12}\left(2\,R_{\rho\lambda}V^{\rho\lambda} - R\,V^{\rho}{}_{\rho}\right)} + \boxed{\frac{1}{48}\left(V^{\rho}{}_{\rho}\,V^{\lambda}{}_{\lambda} + 2\,V_{\rho\lambda}\,V^{\rho\lambda}\right)}\right]$$



HK introduces an unphysical IR regulator [Avramidi & Barvinski, '85]

Local HK expansion insensible to IR $\log(m^2/p^2)$

With Feynman diagrams

Expand around flat background ($\Lambda = 0$) [Julve & Tonin, '78]

$$\bar{g}_{\mu\nu} = \eta_{\mu\nu} + f_{\mu\nu}$$

Fourier space well defined ➡ I can use <u>Feynman diagrams</u>

$O(f^2)$ is enough to reconstruct R^2 terms in Γ_{1-loop}

Sensitive to IR $\log(p^2)$!

$$H = \boxed{\Delta} + \mathcal{D}^{\rho\lambda\mu\nu}\partial_\rho\partial_\lambda\partial_\mu\partial_\nu + \mathcal{C}^{\rho\lambda\mu}\partial_\rho\partial_\lambda\partial_\mu + \mathcal{V}^{\rho\lambda}\partial_\rho\partial_\lambda + \mathcal{N}^{\rho}\partial_\rho + \mathcal{U}$$

Flat propagator

$$Tr\,\log(H) = Tr\,\log(\Delta + A)$$

$$\approx Tr\left[\log\Delta + \cancel{A\frac{1}{\Delta}} - \boxed{\frac{1}{2}A\frac{1}{\Delta}A\frac{1}{\Delta}} + \cdots\right]$$

$O(f^2)$ can be neglected in A

New IR divergent bubble diagrams

If $\Delta = \Box^2$

$$-\frac{1}{2} Tr\left[A\frac{1}{\Box^2}A\frac{1}{\Box^2}\right] = \int d^{4-2\epsilon}q \, \frac{Num}{q^4(q+p)^4}$$

vertices	numerator	$\log(p^2)$ term
$\mathcal{U}\mathcal{U}$	$-\frac{1}{2}\mathcal{U}_{AB}\mathcal{U}^{BA}$	$-\frac{\mathcal{U}^{AB}\mathcal{U}_{BA}}{16\pi^2 p^4}$
$\mathcal{U}\mathcal{N}$	$-\frac{i}{2}\mathcal{U}_{AB}\mathcal{N}^{\mu BA}q_\mu$	$i\frac{\mathcal{U}^{AB}\mathcal{N}^{\mu}_{BA}p_\mu}{32\pi^2 p^4}$
$\mathcal{U}\mathcal{V}$	$\frac{1}{2}\mathcal{U}_{AB}\mathcal{V}^{\mu\nu BA}q_\mu q_\nu$	$\frac{\mathcal{U}^{AB}\mathcal{V}^{\mu\nu}_{BA}p_\mu p_\nu}{32\pi^2 p^4}$
$\mathcal{U}\mathcal{C}$	$\frac{i}{2}\mathcal{U}_{AB}\mathcal{C}^{\mu\nu\rho BA}q_\mu q_\nu q_\rho$	$-i\frac{\mathcal{U}^{AB}\mathcal{C}^{\mu\nu\rho}_{BA}p_\mu p_\nu p_\rho}{32\pi^2 p^4}$
$\mathcal{U}\mathcal{D}$	$-\frac{1}{2}\mathcal{U}_{AB}\mathcal{D}^{\mu\nu\rho\sigma BA}q_\mu q_\nu q_\rho q_\sigma$	$-\frac{1}{32\pi^2}\left(\frac{\mathcal{U}^{AB}\mathcal{D}^{\mu\nu\rho\sigma}_{BA}p_\mu p_\nu p_\rho p_\sigma}{p^4} - \frac{\mathcal{U}^{AB}\mathcal{D}^{\mu}{}_{\mu}{}^{\nu}{}_{\nu BA}}{8}\right)$

Apparently nonlocal contributions!

Localization

$$\mathcal{U} \sim \partial\partial\partial\partial f, \mathcal{N} \sim \partial\partial\partial f, \mathcal{V} \sim \partial\partial f, \mathcal{C} \sim \partial f, \mathcal{D} \sim f$$

$$\frac{\mathcal{U}\mathcal{A}p^{4-\dim[\mathcal{A}]}}{p^4} \sim \frac{p^4 f\, p^4 f}{p^4}$$

"nonlocal logs" become local $+O\left(f^3\right)$ nonlocalities

$$\frac{\mathcal{U}\mathcal{A}p^{4-\dim[\mathcal{A}]}}{p^4} \sim p^2 f\, p^2 f$$

All f^2 terms can be rearranged in C^2 and R^2 up to topological terms

Comparing beta functions

$\boldsymbol{\mu}$-running

$$\beta_\lambda = -\frac{\lambda^2}{(4\pi)^2}\left[\frac{1}{30} + \frac{\xi_1}{24}(\xi_1 - 4) - 2\lambda_1 - \lambda_2\right]$$

$$\beta_\xi = -\frac{\xi^2}{(4\pi)^2}\left[\frac{1}{18} + \frac{\xi_1}{18} + \frac{5\xi_1^2}{72} + \frac{\xi_2}{3} + \frac{\xi_1\xi_2}{3} + \xi_2^2 - \frac{2\lambda_2}{3} - 2\lambda_3\right]$$

$\boldsymbol{p}$-running

$$\beta_\lambda = -\frac{\lambda^2}{(4\pi)^2}\left[\frac{1}{30} + \frac{\xi_1}{24}(\xi_1 - 4)\right]$$

$$\beta_\xi = -\frac{\xi^2}{(4\pi)^2}\left[\frac{1}{18} + \frac{\xi_1}{18} + \frac{5\xi_1^2}{72} + \frac{\xi_2}{3} + \frac{\xi_1\xi_2}{3} + \xi_2^2 - 2\lambda_4^2\right]$$

Discrepancy due to λ_i

Universality with $\lambda_i = 0$ $\Rightarrow$ $U = 0$ $\Rightarrow$ Shift symmetry

Conformal theory: $\lambda_i = 0, \xi_1 = 2, \xi_2 = \frac{3}{2}$

Conformal anomaly $< T > = -\frac{1}{15(4\pi)^2} C^2$ is universal!

Quadratic gravity

$$S_g = \int d^4x\sqrt{-g}\left[+2\Lambda - Z_N R + \frac{1}{2\lambda}C^2 + \frac{1}{\xi}R^2 - \frac{1}{\rho}E\right]$$

$$Z_N = \frac{m_p^2}{16\pi}$$

The Euler density $E = R_{\mu\nu\rho\sigma}^2 - 4\mathrm{R}_{\mu\nu}^2 + \mathrm{R}^2$ is topological in d=4

particle	spin	Mass^2
graviton	2	0
ghost	2	$\lambda\, m_p^2/2$
scalar	0	$-\xi\, m_p^2/12$

Background field method

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$$

$$S = S_g + S_{GF} + S_{FP}$$

$$S = \bar{S}_g + \int d^4x\sqrt{-\bar{g}}\frac{1}{2}h\frac{\delta^2 S_g}{\delta h^2}\Big|_{g_{\mu\nu}=\bar{g}_{\mu\nu}} h + \frac{1}{2a}\int d^4x\sqrt{-\bar{g}}F^{\mu}(h)Y_{\mu\nu}F^{\nu}(h) + \bar{c}^{\mu}Y_{\mu}{}^{\rho}\Delta_{GH\,\rho\nu}c^{\nu} + O\big(h^3\big)$$

$$h\frac{\delta^2(S_g+S_{GF})}{\delta h^2}\Big|_{g_{\mu\nu}=\bar{g}_{\mu\nu}} h = hHh$$

One-loop effective action:

$$\Gamma_{1-loop} = \bar{S}_g + \frac{1}{2}Tr(\log H) - \frac{1}{2}Tr(\log Y) - Tr(\log \Delta_{GH})$$

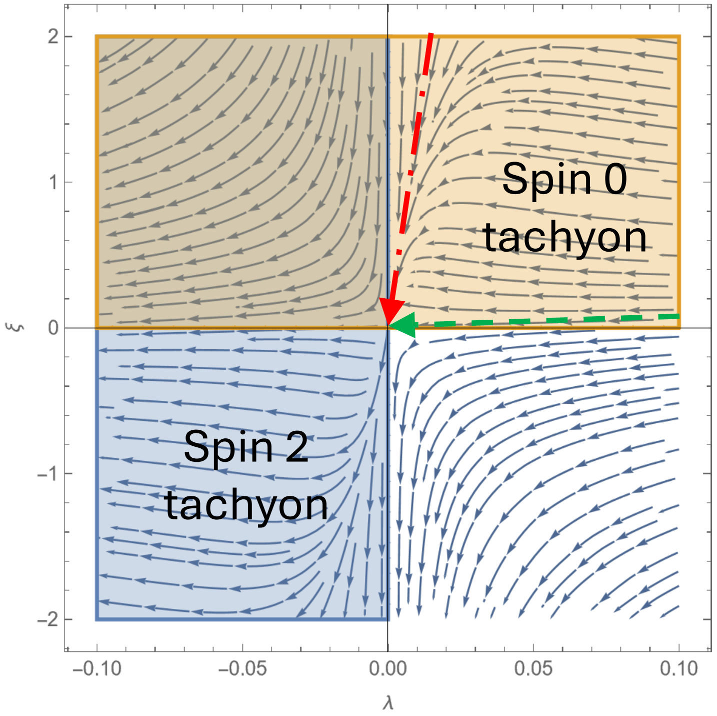
Beta functions of quadratic gravity

$$\omega = -\frac{3\lambda}{\xi}$$

μ-running

$$\beta_\lambda = -\frac{1}{(4\pi)^2}\frac{133}{10}\lambda^2$$

$$\beta_\omega = -\frac{1}{(4\pi)^2}\frac{25 + 1098\omega + 200\omega^2}{60}\lambda$$



$$\lambda^* = 0$$

$$\boxed{\omega_1^* \approx -5.5}$$

$$\boxed{\omega_2^* \approx -0.023}$$

Spin 0 tachyon

p-running

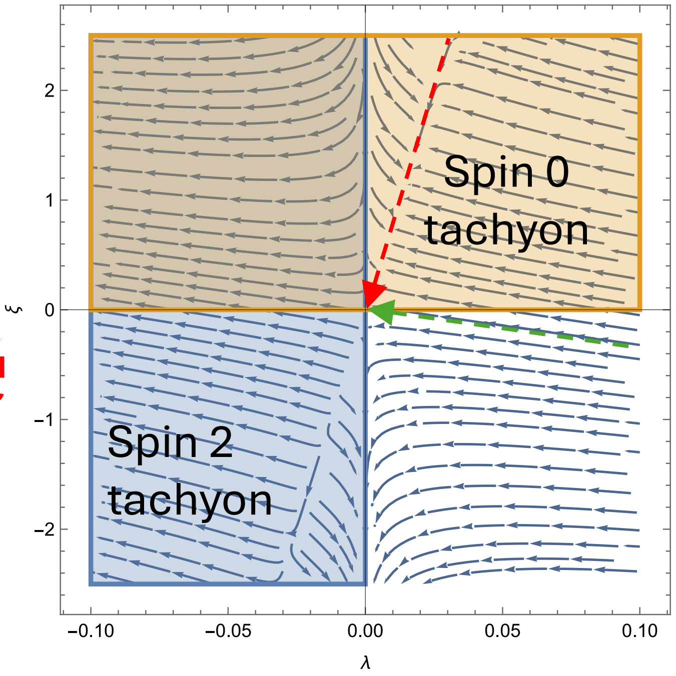
$$\beta_\lambda = -\frac{\lambda^2(539\omega + 20)}{240\pi^2\omega}$$

$$\beta_\omega = \lambda\frac{1400\omega^2 - 1138\omega - 45}{480\pi^2}$$

$$\lambda^* = 0$$

$$\boxed{\omega_1^* \approx 0.85}$$

$$\boxed{\omega_2^* \approx -0.038}$$



Conformal Gravity

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2\lambda}C^2 - \frac{1}{\rho}E\right]$$

Topological terms do not contribute to scattering amplitudes

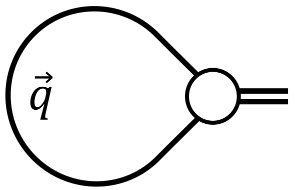
$\beta^{\bar{p}}(\rho)$ not defined

$$\beta_\lambda^{HK} = -\frac{1}{(4\pi)^2}\frac{199}{15}\,\lambda^2 \qquad \beta_\lambda^{\bar{p}} = -\frac{1}{(4\pi)^2}\frac{93}{5}\lambda^2$$

No qualitative differences

Conformal anomaly not well defined with dynamical metric

In $d = 2$



$$\sim \int d^2q \frac{1}{q^2}$$ ➡ log divergent

Potentially the same problem!

CP^1 (or $O(3)$) NLSM

[D.B., John Donoghue, Gabriel Menezes, Roberto Percacci, ’24]

$$S = \frac{1}{2g}\int d^2x \frac{\partial_\mu \phi^a \partial^\mu \phi^a}{\left[1 + \frac{1}{4}\phi^a\phi^a\right]^2} \quad a = 1,2$$

Usually β_g computed from tadpoles [Shifman ‘12,...] unphysical?

$2 \to 2$ amplitude

The simplest scattering process involving g is a $2 \to 2$ process

$$\phi_1 + \phi_1 \to \phi_2 + \phi_2$$

is IR safe at one-loop

$$\mathcal{M} = g^2(\mu)s - \frac{g^4 s}{8\pi}\left[\log\left(-\frac{t}{\mu^2}\right) + \log\left(-\frac{u}{\mu^2}\right)\right] - \frac{g^4}{8\pi}(t-u)\log\left(\frac{t}{u}\right)$$

Logs of the artificial IR cutoff m from tadpole diagrams are cancelled by IR divergent bubble diagrams and replaced with $\log(p)$

$$\beta^{\bar{p}}(g) = \beta^{\mu}(g) = -\frac{g^3}{4\pi}$$

Conclusions

- Renormalization and running are not equivalent concepts
- In higher derivatives theories UV-IR mixing in beta functions
- The heat kernel misses IR running
- Taking into account also IR contributions, there exists a unique AF trajectory without tachyons in quadratic gravity
- In $d = 2$ $CP(1)$ NLSM universality recovered thanks to unitarity (Kinoshita–Lee–Nauenberg theorem). The same for ASQG, if unitary?

Is p-running of couplings "physical"?

$$\delta\Gamma_{(1)} \sim Tr\left[\mathrm{H}^{-1\,\mu\nu\rho\delta\alpha}\epsilon_{\gamma\delta}\nabla^{\gamma}\Delta^{-1}_{\mathrm{gh}\,\alpha\beta}\delta\,\frac{\delta F^{\beta}}{\delta g^{\mu\nu}}\right]+$$

$$Tr\left[\frac{1}{2}\Delta^{-1\,\alpha\beta}_{\mathrm{gh}}\epsilon^{\mu\nu}\nabla_{\mu}\left(\nabla_{(\alpha}g_{\nu)\gamma}-H^{-1\,\rho\sigma\lambda\delta}\epsilon_{\eta\lambda}\nabla^{\eta}\right)\Delta^{-1\,\zeta}_{\mathrm{gh}\quad\delta}\,\delta\left(Y^{-1}_{\zeta\beta}\right)\right],$$ [Avramidi '86]

where $\epsilon_{\mu\nu}$ E.O.M.

In R^2, $R^{\mu\nu}R_{\mu\nu}$, E basis

$$\epsilon_{\mu\nu}=\alpha\left[2RR_{\mu\nu}-2\nabla_{\mu}\nabla_{\nu}R+g_{\mu\nu}\left(2\Box R-\frac{1}{2}R^2\right)\right]$$

$$+\beta\left[2R_{\rho\sigma}R^{\rho\ \ \sigma}_{\ \mu\ \ \nu}-\nabla_{\mu}\nabla_{\nu}R+\Box R_{\mu\nu}+\frac{1}{2}g_{\mu\nu}\left(\Box R-R_{\rho\sigma}R^{\rho\sigma}\right)\right]=0$$

UV divergences proportional to $\epsilon^{\nu}{}_{\nu}=\Box R$

μ-running of $\mathcal{R}^2$ terms is gauge invariant

IR logs can generate R^2, $R^{\mu\nu}R_{\mu\nu}$

p-running can be gauge dependent

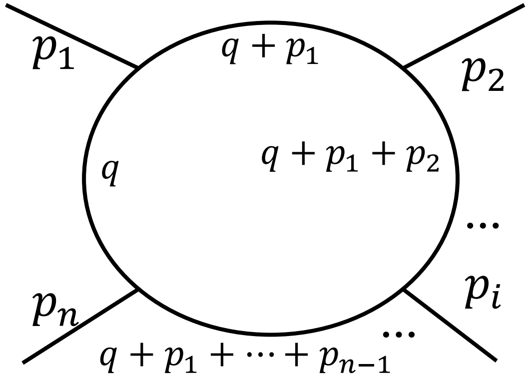
Future perspectives

- Going on-shell with the effective action
- Compute scattering amplitudes/cross sections
- Can the same result be reproduced with non-local HK expansion?
- Do the same IR contributions appear also wit the FRG in the $k \to 0$ limit?

Thank you!

2 derivatives theories

In general $\log(p^2/\mu^2)$ can arrive only from the UV region of loop integrals



$$\sim \int d^4q \, \frac{N(p,q_i)}{q^2(q+p_1)^2 \times \cdots \times (q+p_1+\cdots+p_{n-1})^2}$$

$q \ll p_i$ $\qquad\qquad$ $q \gg p_i$

$$\frac{1}{p_1^2 \times \cdots \times (p_1+\cdots+p_{n-1})^2} \int d^4q \, \frac{N(p,q_i)}{q^2} + \cdots \qquad\qquad \int d^4q \boxed{\frac{N(p,q_i)}{q^{2n}} + \cdots}$$

The UV regions are equal in massive and massless theories, up to $O(m_i/p)$

$$\beta^{\bar{p}}(g) = \beta^{\mu}_{m_i=0}(g) = \beta^{\mu}_{m_i\neq 0}(g) \qquad \text{when} \quad \bar{p} \gg m_i$$