Locally covariant Lorentzian renormalization group

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Università di Milano, funding from the ERC grant FermiMath 101040991

02-04-2025

Quantum spacetime and the renormalization group

Based on joint works with

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Lorentzian challenges

Or: life on the ugly side of AS

Wetterich equation:

$$\partial_k \Gamma_k = \frac{1}{2} \int \partial_k q_k (\Gamma_k^{(2)} - q_k)^{-1}$$

E.g.: $(\Gamma_k^{(2)} - q_k) = \Box_x - q_k, \quad (\Gamma_k^{(2)} - q_k)^{-1} = \text{"}(-p_0^2 + |\vec{p}|^2 + q_k(p))^{-1}\text{"}$

- On general spacetimes: **no** Wick rotation*, heat kernels*, Fourier transforms
- Construction of **interacting propagator** $(\Gamma_k^{(2)} - q_k)^{-1}$:
  Infinite family of fundamental solutions for hyperbolic operators
- Choice of **coarse-graining**:
  - Choice of **ordering**: coarse-grain space-like momenta, time-like momenta,...
  - q_k has to preserve Lorentz invariance, causality, finiteness

Both **conceptual** and **technical** problems

Baldazzi Percacci Skrinjar CQG 2019, *Banerjee Niedermaier 2024, Stromhaier Zelditch 2023

Lorentzian bibliography

- Analytic continuation RG: Floerchinger JHEP 2011, Banerjee Niedermaier 2024...
- Spectral fRG: Fehre Litim Pawlowski Reichert PRL 2021, Pawlowski Reichert 2023, Braun *et al.* SciPost Phys.Core 6 2023, Pastor-Gutiérrez Pawlowski Reichert Ruisi 2024,...
- Spectral geometry: Ferrero Reuter JHEP 2022, Ferrero Ripken SciPost Phys. 2022, ...
- Spatial RG Banerjee Niedermaier Nucl. Phys. B 2022, ...
- Foliated RG: Manrique Rechenberger Saueressig PRL 2011, Rechenberger Saueressig JHEP 2013, Biemans Platania Saueressig JHEP, PRD 2017, Saueressig Wang 2025,...
- Asymptotically safe canonical quantum gravity: Thiemann JHEP 2024, Ferrero Thiemann Universe 2024, 2025,...
- **Locally covariant** RG: Dappiaggi Nava Sinibaldi Rev. Math. Phys. 2025 (with boundaries!),...

Locally covariant FRG

Mathematical formulation:

- Lorentzian from the outset
- Preserves causality, unitarity
- Perturbative construction of observables
- General covariance: RG flow equation takes the same form in all spacetimes

Strategies for our Lorentzian challenges:

- Interacting propagator $(\Gamma_k^{(2)} - q_k)^{-1}$: choice of a reference state, construction from QFTCS, Hadamard regularization
- Coarse-graining: **local** (Callan-Symanzik) regulator
  - Pros: preserves causality and unitarity, acts as a spacetime-dependent mass, extended BRST invariance (regulator in the trivial sector of BRST cohomology)
  - Cons: do not regularize UV divergences, no direct Wilsonian interpretation

Relativistic QFT

"A mathematical rigorous treatment of the FRG: the ugly." – Benjamin Knorr

In curved spacetime: no preferred vacuum $\Leftrightarrow$ no preferred Hilbert space representation $\Rightarrow$ algebraic approach

Quantum theory: A globally hyperbolic spacetime $(\mathcal{M}, g)$, a unital $*-$algebra of local observables $\mathcal{A}(\mathcal{O} \subset \mathcal{M})$, a state $\omega : \mathcal{A} \to \mathbb{C}$

Local interactions: represented as power series in the free algebra $\mathcal{A}[[\lambda]]$

Callan-Symanzik local regulator: $Q_k(\varphi) = -\frac{1}{2} \int_{\mathcal{M}} q_k(x) : \varphi^2(x) :$
Perturbative definition Effective average action: $\Gamma_k(\phi) := \omega(O) \in \mathbb{C}[[\lambda]]$

$$\Rightarrow \textbf{ Interpolation: } \Gamma \xleftarrow{k \to 0} \Gamma_k \xrightarrow{k \to \infty} S(\phi) + C$$

Haag Kastler (1964), Brunetti Fredenhagen Dütsch (2000, 2009); Brunetti Fredenhagen Verch (2001); Hollands Wald (2001, 2002); Yngvason (2004); Fredenhagen Rejzner (2012, 2013);...
Reviews: Rejzner (2016), Hollands Wald (2014), Advances in AQFT (2015)

RG flow equations

"He must, so to speak, throw away the ladder after he has climbed up it." – Wittgenstein

$$\partial_k \Gamma_k = \frac{i}{2} \int_{\mathcal{M}} \partial_k q_k(x) : G_k : (x, x)$$

$$\left(\Gamma_k^{(2)} - q_k \right) G_k = -1$$

Finiteness

- Local regulator $q_k \in C_c^\infty(\mathcal{M}) \longrightarrow$ IR finite
- Normal-ordering $\longrightarrow$ UV finite

Key features

- State dependence
- Normal ordering introduces additional parameter α

Nonperturbative definition: EAA is a 1-parameter family

$$\Gamma_k : \phi \mapsto \Gamma_k(\phi) \in \mathcal{F}(\mathcal{E}_{\text{mean}})(\mathcal{M}), \; k \in \mathbb{R}^+$$

Interacting propagator

Which G_k?

$(\Gamma_k^{(2)} - q_k)G_k = -1$

Lorentzian spacetime:

$\Gamma_k^{(2)} - q_k$ hyperbolic $\Rightarrow$ infinite family of inverses!

Idea: fix choice of interacting propagator by fixing a free state

- DSE:

$$\Gamma_k^{(1)}(\phi) = S_0^{(1)}(\phi) + \langle V^{(1)}(\varphi)\rangle \Rightarrow$$

- EAA decomposition:

$$\Gamma_k^{(2)}(\phi) - q_k := S_0^{(2)} + U_k^{(2)}(\phi)$$

$\Rightarrow$ Construct G_k from the free Feynman propagator, perturbatively in $U_k^{(2)}$

Interacting propagator

Solve the free wave equation $S^{(2)}\Delta_F = \delta \Leftrightarrow$ choose **free** Hadamard Feynman propagator Δ_F, then

- In the free case $U_k = 0 \Rightarrow -i :G_k: = :\Delta_F: \in C^\infty$
- $:\Delta_F: = \Delta_F - h_F$, with $h_F =$ UV singularity of the vacuum state

$$h_F(x,y) \propto \frac{u}{|x-y|} + v\log(|x-y|\alpha)$$

- In the interacting case

$$\begin{aligned} -i :G_k: &= (1-\Delta_R^U U_k^{(2)}) :\Delta_F: (1-U_k^{(2)}\Delta_A^U) \\ &= (1-i\Delta_F U_k^{(2)})^{-1} :\Delta_F: \end{aligned}$$

$H_F = (1-\Delta_R^U U_k^{(2)})h_F(1-U_k^{(2)}\Delta_A^U)$, Hadamard singularity in the LPA.

Local solutions

Dyson series

$$\partial_k U_k = -\frac{1}{2}\int_{\mathcal{M}} \partial_k q_k \sum_n (i\Delta_F U_k^{(2)})^n : \Delta_F :$$

Any possible interaction is generated along the RG flow

Loss of derivatives: RHS depends on the **inverse** $(1 - i\Delta_{F,k}U_k^{(2)})^{-1} \Rightarrow$ the fundamental solution of the RG flow generically depends on $U_k^{(2)} \Rightarrow$
Fixed-point iteration technique in any C^n fails
$\longrightarrow$ **Nash-Moser theorem** in space of C^∞−functions

Theorem (ED, Pinamonti (2024)**)**

For scalar fields in the LPA on static spacetimes, unique local solutions of the RG flow equations exist.

N.B.: Not all states admit local solutions $\Rightarrow$ RG selects admissible states

Example: de Sitter quantum gravity

Choice of state $\Leftrightarrow$ Choice of background: **Explicit** expression of : G_k : depends on the choice of a background spacetime
But in de Sitter: **unique** dS-invariant Hadamard ground state

Main problem: construction of : G_k : for massive gravitons and ghosts in a general gauge:

$$(P^{\mu\nu\alpha\beta}_{\xi,\zeta})G_{k,\alpha\beta\rho'\sigma'} = \delta(x,x')g^{\mu}{}_{(\rho'}g^{\nu}{}_{\sigma')}$$

Interacting propagator

$$
\begin{aligned}
P^{\rho\sigma\mu\nu}_{\xi,\zeta} \equiv \frac{1}{2}\left[g^{\rho(\mu}g^{\nu)\sigma} - \frac{1}{2}\left(2-\frac{1}{\xi\zeta^2}\right)g^{\rho\sigma}g^{\mu\nu}\right]\Box \\
-\left(1-\frac{1}{\xi}\right)\nabla^{(\rho}g^{\sigma)(\mu}\nabla^{\nu)} + \frac{1}{2}\left(1-\frac{1}{\xi\zeta}\right)\left(g^{\mu\nu}\nabla^{\rho}\nabla^{\sigma}+g^{\rho\sigma}\nabla^{\mu}\nabla^{\nu}\right) \\
-\frac{m^2+2H^2}{2}g^{\rho(\mu}g^{\nu)\sigma} + \frac{M^2+m^2-4H^2}{8}g^{\rho\sigma}g^{\mu\nu}\,.
\end{aligned}
$$

Gauge-adapted regulator:

$$
q_k^{\rho\sigma\mu\nu} = \left[\bar{g}^{\rho(\mu}\bar{g}^{\nu)\sigma} - \frac{1}{2}\left(2-\frac{1}{\xi\zeta^2}\right)\bar{g}^{\rho\sigma}\bar{g}^{\mu\nu}\right]k^2\,.
$$

$$
\Rightarrow m^2 = k^2 + 2(3H^2-\Lambda_k)\,, \quad M^2 = k^2\left(3-\frac{2}{\xi\zeta^2}\right) + 2(3H^2-\Lambda_k)
$$

Interacting propagator

Recipe

- Use dS invariance: $G_{k,\alpha\beta\rho'\sigma'}(x,x') = G_{k,\alpha\beta\rho'\sigma'}(Z)$
- Make the most general dS invariant ansatz
- Fix free parameters by requiring Hadamard form, finite massless limit
- Expand $x' \to x$, identify Hadamard singularity, subtract the Hadamard parametrix (introducing a free Hadamard parameter α)
- Plug : G_k : into RG and run (the flow)
- Identify dependence on the gauge and Hadamard parameters of the fixed points and critical exponents

Phase portrait

RG flow for the $\beta-$functions of the running Newton's and cosmological constants

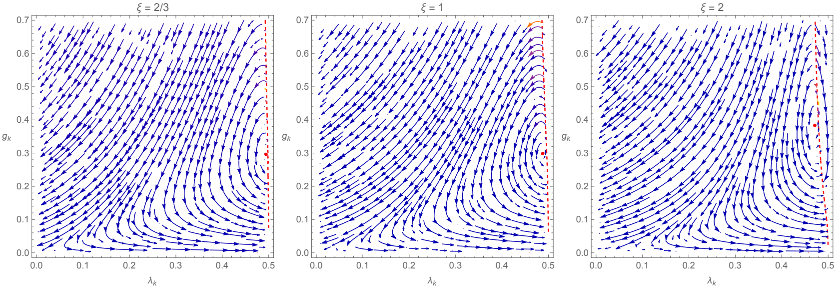

Figure 1: Flow diagram with $\zeta = \frac{1}{2}$ and $\alpha = \frac{2}{5}$, for different values of ξ.

UV fixed point

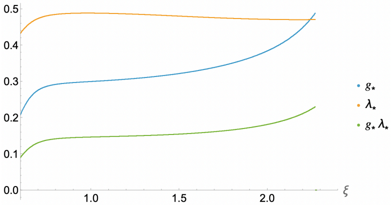

Figure 2: UV fixed point with $\zeta = \frac{1}{2}$ and $\alpha = \frac{2}{5}$ for different values of ξ.

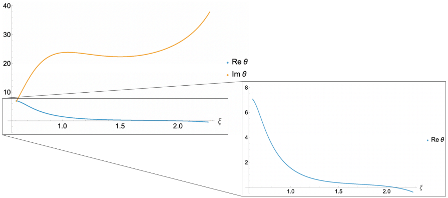

Figure 3: Critical exponents with $\zeta = \frac{1}{2}$ and $\alpha = \frac{2}{5}$ for different values of ξ.

Outlook

Key takeaways:

- Give me a Hadamard propagator and I can give you the Lorentzian flow
- Lorentzian FRG: not as ugly as it may seem!

Future directions

1. Beyond LPA: G_k = Hadamard Green function for **higher order** Green differential operators
2. **Global** solutions: RG trajectories from Nash-Moser
3. **Essential** RG: can we remove dependence on gauge **and** Hadamard parameter?
4. **Applications:** Flow of observables: amplitudes, cosmological relational observables, precision computations,...

Thank you for your attention!