Asymptotic Safety within on-shell perturbation theory

based on work with Renata Ferrero

2411.00938 [hep-th]

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3rd April 2025

Introduction

- Weinberg, Parisi: Einstein's gravity, treated as a quantum field theory of the metric tensor, could be ultraviolet complete due to a suitable interacting fixed point of the renormalisation group (RG)

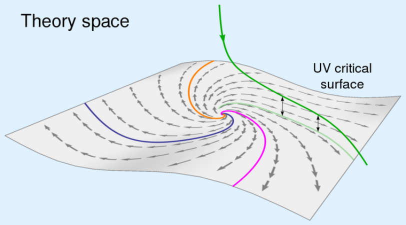

Figure: Theory space with a UV fixed point and its UV critical surface

Introduction

- Asymptotic safety requires a fixed point for **essential** couplings that enter observables such as scattering amplitudes

- Physics is **independent** of the way we **parameterise** the metric fluctuations and fix the **gauge**

$$\phi_{\mu\nu} = g_{\mu\nu}\,, \qquad \tilde{\phi}^{\mu\nu} = \sqrt{g}g^{\mu\nu}$$

$$\partial_\mu g^{\mu\nu} = 0\,, \qquad \partial_\mu(\sqrt{g}g^{\mu\nu}) = 0$$

- However, non-perturbative results can to depend on these unphysical choices (Benedetti,Nink, Gies, Knorr, Lippoldt, Ohta, Percacci, Pereira,Vacca, KF)

- Goal: Find a scheme where these dependencies are absent.

Road map

How? Remove the dependencies in a controlled manner.

- Fix the flow of inessential couplings that do not enter scattering amplitudes.
- Implement a generalisation of dimensional regularisation.
- Use a RG improved perturbation theory at one-loop.

First approaches

- First approach to asymptotic safety: start in $d = 2 + \varepsilon$ dimensions:

$$\Gamma_{\text{div}} \sim \frac{\mu^{\varepsilon}}{\varepsilon} \int d^d x \sqrt{g} R(g)$$

- One loop beta function

$$\beta_{\tilde{G}} = \varepsilon \tilde{G} - b \tilde{G}^2$$

- IR fixed point: $\tilde{G} = 0$

- UV fixed point $\tilde{G}_* = \varepsilon / b$.

First approaches

- Problem (Weinberg, Christensen, Duff, Gastmans, Kallosh,Truffin): b **depends** on the **gauge** and on the choice of **parameterisation** of the metric (e.g. $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ or $g^{\mu\nu} = \bar{g}^{\mu\nu} + h^{\mu\nu}$)

- Really:

$$\Gamma_{\mathrm{div}} = A\frac{\mu^{\varepsilon}}{\varepsilon}\int d^d x\sqrt{g}\Lambda_{\mathrm{cc}} + B\frac{\mu^{\varepsilon}}{\varepsilon}\int d^d x\sqrt{g}R(g)$$

- Solution (H. Kawai and M. Ninomiya 89): If we use a field redefinition of the metric

$$g_{\mu\nu} \to Z_k g_{\mu\nu}$$

to keep the cosmological constant unrenormalised

$$b = \frac{38}{3}$$

independent of the gauge or parameterisation.

Inessential couplings

- We are free to couple the source to different field variables

$$\mathcal{Z}[J] = \int d\hat{\chi} \mathrm{e}^{-S[\hat{\chi}]+\int_x J(x)\hat{\phi}[\hat{\chi}](x)}$$

- If we make a small change in the change of variables

$$\hat{\phi} \to \hat{\phi} + \epsilon\hat{\Phi}$$

the effective action changes by

$$\Gamma \to \Gamma + \epsilon \int_x \Phi(x) \frac{\delta}{\delta\phi(x)}\Gamma\,, \quad \Phi(x) = \langle \Phi(\hat{x}) \rangle$$

Inessential couplings

- One can show that **amplitudes** are **independent** of ζ

$$\frac{\partial}{\partial\zeta}\mathcal{A} = 0$$

whenever

$$\frac{\partial}{\partial\zeta}\Gamma = \int_x \Phi(x)\frac{\delta}{\delta\phi(x)}\Gamma$$

- We call ζ an **inessential coupling** (Weinberg) and the operator on the RHS a **redundant operator** (Wegner).
- Wegner showed **essential scaling exponents** θ are independent of ζ

$$\frac{\partial}{\partial\zeta}\theta = 0$$

- Critical exponents for redundant can be chosen by a choice of scale dependent field redefinition.

Essential RG

- Use RG scale-dependent field redefinitions (Pawlowski '07) to remove inessential couplings from the set of flowing couplings (Baldazzi,Ben Alí Zinati, KF 2105.11482),

- Minimal essential scheme for gravity (Baldazzi, KF 2107.00671): fix the vacuum energy and set all couplings to operators proportional the Ricci tensor to zero (apart from $\frac{1}{16\pi G}$).

$$S = S_{EH+} + \int_x (\cancel{\#\sqrt{g}R^2} + \cancel{\#\sqrt{g}R_{\mu\nu}R^{\mu\nu}} + \cancel{\#\sqrt{g}R^3} + \#C_{\mu\nu\rho\lambda}C^{\rho\lambda\alpha\beta}C_{\alpha\beta}{}^{\mu\nu})$$

- Using the minimal essential scheme in perturbation theory is our first ingredient.

Functional Perturbative methods in $d = 4$ dimensions

- In $d = 4$ dimensions the pole in $1/(d-2)$ appears as a coefficient of $\mu^2\sqrt{g}R$ (and $\mu^2\sqrt{g}\Lambda_{cc}$).

- At one-loop there are poles $1/(d-d_c)$ in all even dimensions $d_c = 0, 2, 4, 6, ..., \infty$ which appear in the coefficients of increasingly higher order operators.

$$\Gamma_{\text{div}} = \sum_{d_c} \frac{\mu^{d-d_c}}{d-d_c} O_{d_c}$$

- This allows us to effectively work functionally i.e. we can keep track of all local operators in the effective action when studying the RG flow.

- This idea was put forward by Weinberg in the initial works but only studied recently (Yannick Kluth 2024).

Functional Perturbative methods in $d = 4$ dimensions

Scheme (at one-loop):

- Retain $g^{\mu}_{\mu} = d$ dependence in divergent parts (Martini, Ugolotti, Del Porro, Zanusso).
- Subtract all poles of the form $1/(d - d_c)$ (within some approximation e.g. up to some $d_{c,\max}$).

Functional Perturbative methods in $d = 4$ dimensions

Advantages:

- Regulator preserves diffeomorphism (BRST) invariance.
- Can easily keep track of unphysical dependence on gauge and parameterisation choices (and remove them!).
- (One)-loop equations can be RG improved giving some non-perturbative information
- connection to "proper time flows" via retaining "$g^{\mu}_{\mu} = d$ dependence" aka Zanusso et al non-minimal subtraction.

Summary: One has a powerful scheme without unphysical dependencies.

Einstein-Hilbert+ approximation

- Einstein-Hilbert+ action

$$S_{\rm EH+} = \int d^4x\sqrt{\det g}\,\frac{1}{16\pi G}(-R + 2\Lambda_{\rm cc}) + S_{\rm gf} + S_{\rm gh} + \text{topological term}$$

Later I will use ρ for the cc where $\Lambda_{\rm cc} = G\rho$.

$$\text{topological term} = \vartheta\int_x \sqrt{g}\,\mathfrak{E} = \vartheta\int_x \sqrt{g}(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\rho\sigma\mu\nu}R^{\rho\sigma\mu\nu})$$

Einstein-Hilbert+ approximation

- Two running couplings Newton's constant and the cosmological constant.

$$\Gamma = S + \frac{1}{2}\mathrm{Tr}\log\Big[K^{-1}(S^{(2)} + S^{(2)}_{\mathrm{gf}})\Big] - \mathrm{Tr}[\mathcal{Q}_{\mathrm{FP}}]\,.$$

Measure contribution:

$$K^{\mu\nu,\rho\lambda}(x,y) = \frac{\sqrt{g}}{64\pi G}(g^{\mu\rho}g^{\nu\lambda} + g^{\mu\lambda}g^{\nu\rho} - g^{\mu\nu}g^{\rho\lambda})\delta(x-y)$$

- Evaluate the trace using the heat kernel expansion at $g = \bar{g}$ involves an expansion in curvature.

Divergencies

- At one-loop and EH+ approximation: using field redefinitions $\iff$ using equations of motion $\iff$ Use Einstein equations

$$\Gamma = \int_x \sqrt{g}\left(\frac{a_0(d)}{d}\mu^d + \frac{a_2(d)}{d-2}\mu^{d-2}R + \frac{a_4(d)}{d-4}\mu^{d-4}G\rho R + \frac{a_4'(d)}{d-4}\mu^{d-4}\mathfrak{E}\right) + \dots$$

One then finds the following coefficients

$$a_0(d) = \frac{1}{(4\pi)^{\frac{d}{2}}}\frac{1}{2}(d-3)d$$

$$a_2(d) = \frac{1}{(4\pi)^{\frac{d}{2}}}\frac{1}{12}\left(d^2 - 3d - 36\right)$$

$$a_4(d) = \frac{1}{(4\pi)^{\frac{d}{2}}}\frac{d^3 + 19d^2 - 566d + 1200}{120(d-2)}$$

$$a_4'(d) = \frac{1}{(4\pi)^{\frac{d}{2}}}\frac{1}{360}\left(d^2 - 33d + 540\right).$$

Minimal subtraction

- Each $a_{d_c}(d)$ is a function of the dimension. However

$$\frac{a_{d_c}(d)}{d-d_c} = \frac{a_{d_c}(d_c)}{d-d_c} + \text{finite terms}\ ,$$

- thus a standard minimal subtraction would set the counterterms to

$$S_{\rm ct} = -\int_x \sqrt{g}\left(\frac{a_0(0)}{d}\mu^d + \frac{a_2(2)}{d-2}\mu^{d-2}R + \frac{a_4(4)}{d-4}\mu^{d-4}G\rho R + \frac{a_4'(4)}{d-4}\mu^{d-4}\mathfrak{E}\right)\ .$$

- Note $a_0(0) = 0$ vanishes.

Subtraction

- Using the non-minimal subtraction we subtract the the poles with d dependent coefficients: only $\frac{1}{(4\pi)^{\frac{d}{2}}} \to \frac{1}{(4\pi)^{\frac{d_c}{2}}}$
- Thus we subtract

$$S_{\rm ct} = -\int_x \sqrt{g}\left(\bar{a}_0(d)\frac{\mu^d}{d} + \bar{a}_2(d)\frac{\mu^{d-2}}{d-2}R + \bar{a}_4(d)\frac{\mu^{d-4}}{d-4}G\rho R + \bar{a}_4'(d)\frac{\mu^{d-4}}{d-4}\mathfrak{E}\right)\,.$$

with

$$\bar{a}_0(d) = \frac{1}{2}(d-3)d$$

$$\bar{a}_2(d) = \frac{1}{(4\pi)}\frac{1}{12}\left(d^2-3d-36\right)$$

$$\bar{a}_4(d) = \frac{1}{(4\pi)^2}\frac{d^3+19d^2-566d+1200}{120(d-2)}$$

$$\bar{a}_4'(d) = \frac{1}{(4\pi)^2}\frac{1}{360}\left(d^2-33d+540\right)\,.$$

Beta function

- We adopt the *minimal essential scheme* which means that we fix all dimensionless inessential couplings to their values at the GFP: i.e. to zero with the exception of $\tilde{\rho} = \rho\mu^{-d}$ which is fixed to
$$\frac{\tilde{\rho}}{8\pi} \overset{!}{=} \frac{\bar{a}_0}{d} = \frac{1}{2}(d-3)\,.$$
- Then the beta function for the essential coupling (G in units of ρ) formed by
$$\eta \equiv G\left(\frac{\rho}{4\pi(d-3)}\right)^{\frac{d-2}{d}} \overset{!}{=} G\mu^{d-2},,$$
is given by
$$\beta_\eta = (d-2)\eta + \frac{1}{3}((d-3)d-36)\eta^2 + \frac{(d-3)(d(d(d+19)-566)+1200)\eta^3}{30(d-2)}\,.$$

Critical Exponent

- For $d = 4$ this beta function has a fixed point at $\eta_\star = \frac{1}{87}\left(\sqrt{2905} - 40\right) \approx 0.16$

- the critical exponent is given by

$$\theta = -\left.\frac{\partial \beta_\eta}{\partial \eta}\right|_{\eta=\eta_\star} = \frac{4}{261}\left(581 - 8\sqrt{2905}\right) \approx 2.296\,.$$

- This value is numerically in good agreement with calculations using the functional renormalisation group in various approximations which also use the MES (Baldazzi, Falls, Kluth, Knorr).

$$\theta_{\mathrm{FRG}} \approx 2.2$$
$$\theta_{\mathrm{DR}} \approx 2.3$$

Critical Exponent

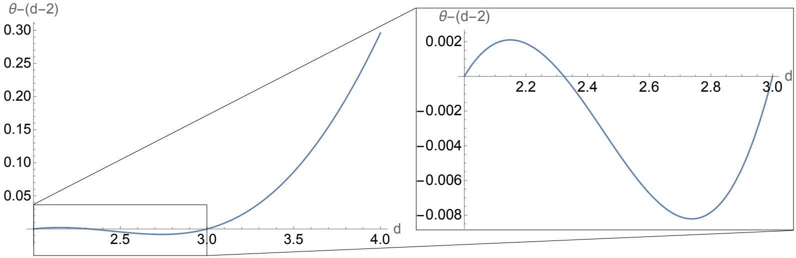

Figure: Plot of the critical exponent as a function of the dimension.

Proper time flows

- It turns out that the scheme we are using at one-loop is equivalent a *generalised proper-time flow equation*

$$k\partial_k\Gamma_k[\phi] = -\Psi_k[\phi]\cdot\frac{\delta}{\delta\phi}\Gamma_k[\phi] + \mathrm{STr}\,\exp(-K_k^{-1}\Gamma_k^{(2)}[\phi]\,k^{-2})\,.$$

- The first term allows for field redefinition to be implemented.
- One can show that the beta functions of the proper time equation and the DR scheme are the equal after a rescaling of couplings $\lambda_i \to (4\pi)^{-d_i}\lambda_i$ e.g.

$$\tilde{G}_{(\mathrm{PT})} \to (4\pi)^{d/2-1}\tilde{G}_{(\mathrm{DR})}, \quad \tilde{\rho}_{(\mathrm{PT})} \to (4\pi)^{-d/2}\tilde{\rho}_{(\mathrm{DR})}\ ,$$

- Proper time flows give very good estimates for critical exponents in scalar field theories (Bonanno, Zappalá, Mazza) BUT they are aren't exact (Litim, Pawlowski, Wetterich).

Proper time flows: testing parameterisation (in)dependence

- The degree to which the beta functions obtained from the proper time flow equation depends on the gauge and the parameterisation depends on the approximation used to solve the flow equation.

- We test to see which approximations are independent of these choices by using a general parameterisation of the metric

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} + \frac{1}{2}\left(\tau_1 h_{\mu\rho}h^{\rho}_{\nu} + \tau_2 h h_{\mu\nu} + \tau_3 \bar{g}_{\mu\nu} h_{\rho\sigma}h^{\rho\sigma} + \tau_4 \bar{g}_{\mu\nu} h^2\right) + O(h^3)\,.$$

- Therefore physics should not depend on the parameters τ_i.

Proper time flows: Order $\mathrm{curvature}^2$

- Taking into account the ghosts and exploiting the background field method, the generalised proper-time flow equation for gravity takes the form

$$\left(k\partial_k + \int_x \Psi_{\mu\nu}\frac{\delta}{\delta g_{\mu\nu}}\right)\Gamma = \mathrm{Tr}\, e^{-K^{-1}(\Gamma^{(2)}+S^{(2)}_{\mathrm{gf}})k^{-2}} - 2\mathrm{Tr}\, e^{-\mathcal{Q}_{\mathrm{FP}}[\phi]k^{-2}} \,,$$

  where if we truncate in up to curvature squared terms we take

$$\Psi^g_{\mu\nu}[g] = \gamma_g g_{\mu\nu} + \gamma_R R g_{\mu\nu} + \gamma_{Ricci} R_{\mu\nu} \,,$$

$$\Gamma^{(2)}{}_k{}^{\mu\nu,\rho\lambda}(x,y) = \frac{\delta\Gamma_k}{\delta h_{\mu\nu}(x)\delta h_{\rho\lambda}(y)} \,,$$

  and the equations of motion (Einstein equation)

$$\frac{\delta\Gamma}{\delta g_{\mu\nu}} = \frac{\sqrt{g}}{2}\frac{\rho}{8\pi}g^{\mu\nu} + \frac{\sqrt{g}}{16\pi G}\left(R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu}\right).$$

- One solves for the beta functions of the essential couplings and for the gamma functions which replace the beta functions of the inessential couplings.

Proper time flows: Order $\mathrm{curvature}^2$

- If we only truncate the equation at order curvature squared we obtain beta functions that depend on the parameterisation!

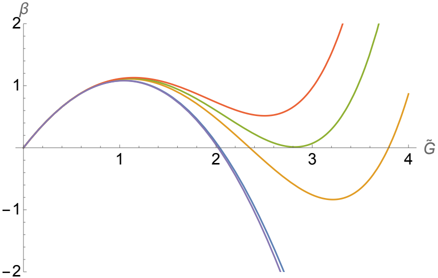

Figure: The beta function for Newton's constant $\beta(g)$ for different parameterisations. The top curve (red) corresponds to $\tau_1 = -0.8$ which does not feature a fixed point. The second highest curve (green) is for $\tau_1 = -0.705$: in this case two fixed points collide. The third curve (orange) corresponds to $\tau_1 = -0.6$ and the two lowest curves are for $\tau_1 = 0$ (blue) and $\tau_1 = 1$ (purple).

Consistent approximations

- However, if we expand the beta function to order G^3 we obtain the same beta function as within dimensional regularisation independent of the parameterisation.
- This teaches us an important lesson: if we want results independent of the gauge we must expand in both the curvature and the Newton's constant.
- If we work to finite order N in curvature we should approximate the beta function for $\tilde{G}$ by its expansion to order $N+1$. Otherwise we get unphysical dependencies.
- Consistent approximation

$$\mathcal{L} = \# + \cdots + \# R^N$$

$$\beta_{\tilde{G}} = (d-2)G + \cdots + \# G^{N+1}$$

Symmetric Spacetimes

- In order to go to all orders in curvature and hence obtain an all orders beta function that is independent of the parameterisation we specialise to symmetric spacetimes so that the only curvature invariants are functions $\int_x \sqrt{g} f(R)$.
- we now write the RG kernel as

$$\Psi_{\mu\nu} = \gamma(R) g_{\mu\nu} \ .$$

- So we will recover the beta function for the essential coupling $\tilde{G}$ and the function $\gamma(R)$ which removes the off-shell terms from the flow equation.
- For now, we will neglect the topological term in the action such that $\vartheta_k = 0$.

Symmetric Spacetimes

- In this case the flow equation can be written in the form

$$\int \mathrm{d}^d x\sqrt{g}\left[\frac{k^d\beta_{\tilde{\rho}}}{8\pi}+\frac{k^{d-2}\beta_{\tilde{G}}R}{16\pi\tilde{G}^2}-k^{d-2}\frac{(d-2)(\gamma(R)+2)\left(R-\frac{2dG_k\rho_k}{d-2}\right)}{32\pi\tilde{G}}\right]=$$
$$=\mathrm{Tr}_0[e^{-\frac{\Delta_0}{k^2}}]+\mathrm{Tr}_{2T}[e^{-\frac{\Delta_2}{k^2}}]-2\mathrm{Tr}_1[e^{-\frac{\Delta_1}{k^2}}]\,,$$

- Above

$$\Delta_0=-\nabla^2-\frac{2R}{d}+\sigma\left(R-\frac{2dG_k\rho_k}{d-2}\right) \quad (1)$$
$$\Delta_1=-\nabla^2-\frac{R}{d} \quad (2)$$
$$\Delta_2=-\nabla^2+\frac{2R}{(d-1)d}+\tau\left(R-\frac{2dG_k\rho_k}{d-2}\right)\,, \quad (3)$$

and the parameters

$$\sigma=\left(\frac{2\left(\tau_1-1\right)}{d}+2d\tau_4+2\tau_2+2\tau_3+1\right)\,, \quad (4)$$
$$\tau=-\frac{(d-2)\left(d\tau_3+\tau_1-1\right)}{d}\,. \quad (5)$$

Symmetric Spacetimes

- Following the MES we put $\beta_{\tilde{\rho}} = 0$ and set $\tilde{\rho} = \tilde{\rho}_{\mathrm{GFP}}$.
- We can solve the flow equation for all R in the range $-\infty < R < \infty$ to obtain both beta function $\beta_{\tilde{G}}$ and $\gamma(R)$.
- Assuming $\gamma(R)$ is regular at the point when the equations of motion

$$R = \frac{2d}{d-2} G_k \rho_k = \frac{2d}{d-2} k^d G \tilde{\rho}_{\mathrm{GFP}}$$

  are satisfied, we see that we can obtain $\beta_{\tilde{G}}$ independently of $\gamma(R)$ which is multiplied by the equations of motion.
- The dependence on the τ_i is also proportional to the equations of motion so $\beta_{\tilde{G}}$ is independent of the parameterisation to all orders in $\tilde{G}$.

Symmetric Spacetimes+

- We can also take into account the topological term in a minimal way by using its beta function obtained from previous approximation.
- In this case we have an extra term

$$k\partial_k\vartheta_k \int_x \sqrt{g}\mathfrak{E} \to k\partial_k\vartheta_k \int_x \sqrt{g}\frac{(d-3)(d-2)R^2}{(d-1)d} \ .$$

- The beta function of the topological coupling is

$$k\partial_k\vartheta_k = k^{d-4}\frac{1}{45}2^{-d-3}\left(d^2-33d+540\right)\pi^{-d/2}+O(G_k) \ ,$$

- Then up to order G^3 we obtain the same beta function as before.

Symmetric Spacetimes+

- We can also evaluate the traces using the heat kernel expansion keeping terms up to R^N and then expanding the beta function for $\tilde{G}$ to order $\tilde{G}^{N+1}$.
- At order $N = 5$ and for $d = 4$ the beta function is

$$\beta_{\tilde{G}} = 2\tilde{G} - \frac{8\tilde{G}^2}{3\pi} - \frac{29\tilde{G}^3}{40\pi^2} - \frac{2459\tilde{G}^4}{68040\pi^3} + \frac{5441\tilde{G}^5}{3265920\pi^4} + \frac{39059\tilde{G}^6}{53887680\pi^5} + O\left(\tilde{G}^7\right) . \quad (6)$$

- At each order from $N = 1$ to $N = 5$ we find a fixed point and the critical exponent converges rapidly to the value at $N = \infty$ as can be seen in table 1.

$O(R^N)$	θ
R	2
R^2	2.296
R^3	2.312
R^4	2.312
R^5	2.311
R^∞	2.311

Table: Critical exponent at every order of the R^N expansion.

Conclusions

- New approach to studying asymptotic safety in perturbation theory
- Combines a new subtraction scheme with the minimal essential scheme
- Parametrization dependence disappears order by order in a double expansion
- More evidence for only one relevant essential coupling

To do list

- Matter fields
- Higher order operators e.g. $C^3_{\mu\nu\rho\lambda}$
- Gauge dependence
- Two-loops: Two-loop corrections to proper time flows (Wetterich)
- Agreement with lattice quantum gravity (?)