Fixed Points of Quantum Gravity from Dimensional Regularization

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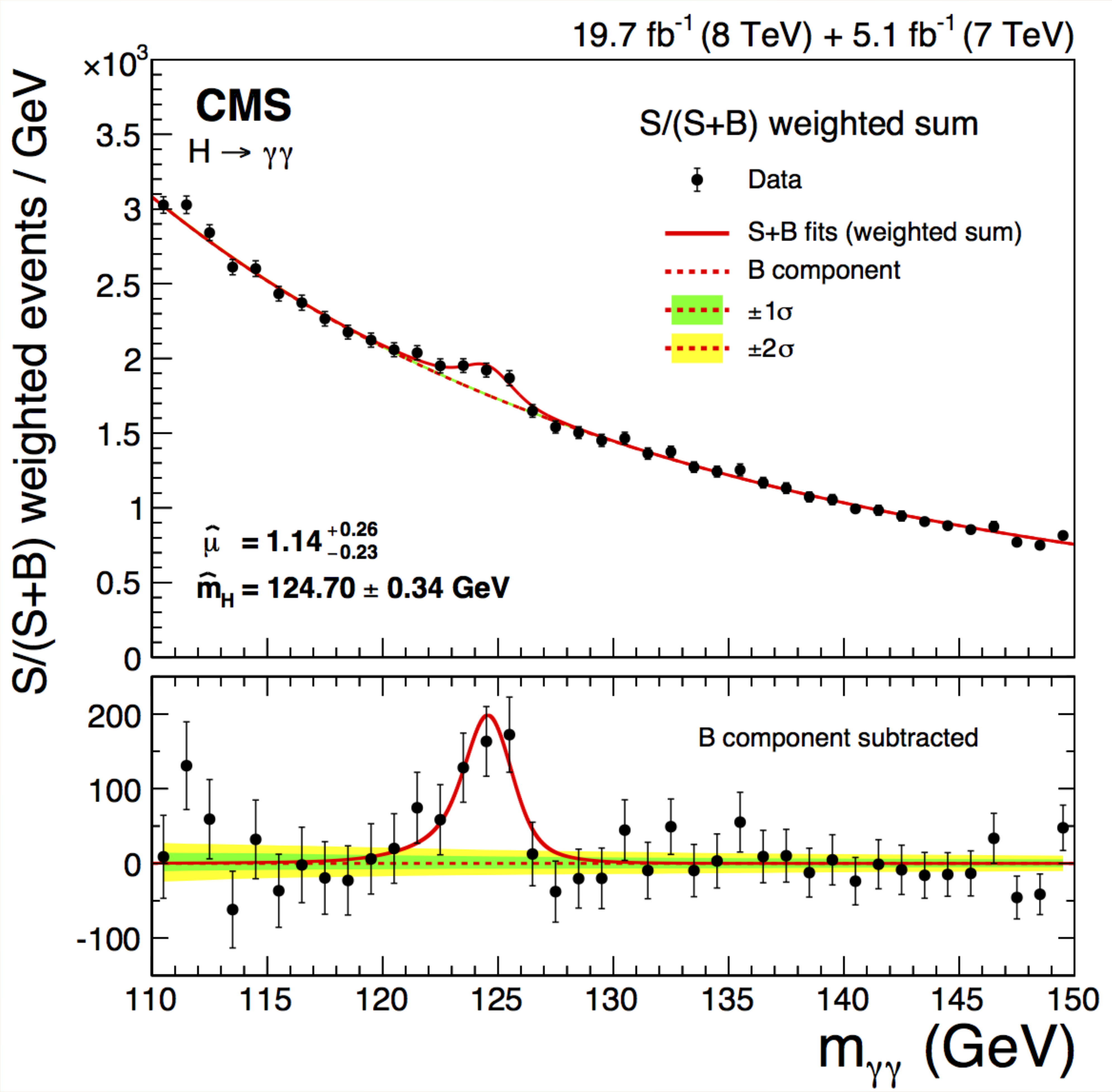
Why care about Perturbation Theory?

- Well understood toolkit to study QFT

- Could be useful to understand open challenges of Asymptotic Safety

  - Lorentzian signature?
  - Diffeomorphism Symmetry?
  - Unitarity?



A Perturbative Approach to Quantum Gravity

- Gravity is not power-counting renormalizable
  - We need to include an infinite number of operators…

$$S = \int \mathrm{d}^4x \sqrt{-g} \left[\frac{R}{16\pi G_0} - \frac{\Lambda_0}{8\pi G_0} + \lambda_E^{(0)} E + \lambda_{C^3}^{(0)} C^3 + \dots \right]$$

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Could be as small as 10^{-7} at the fixed point!

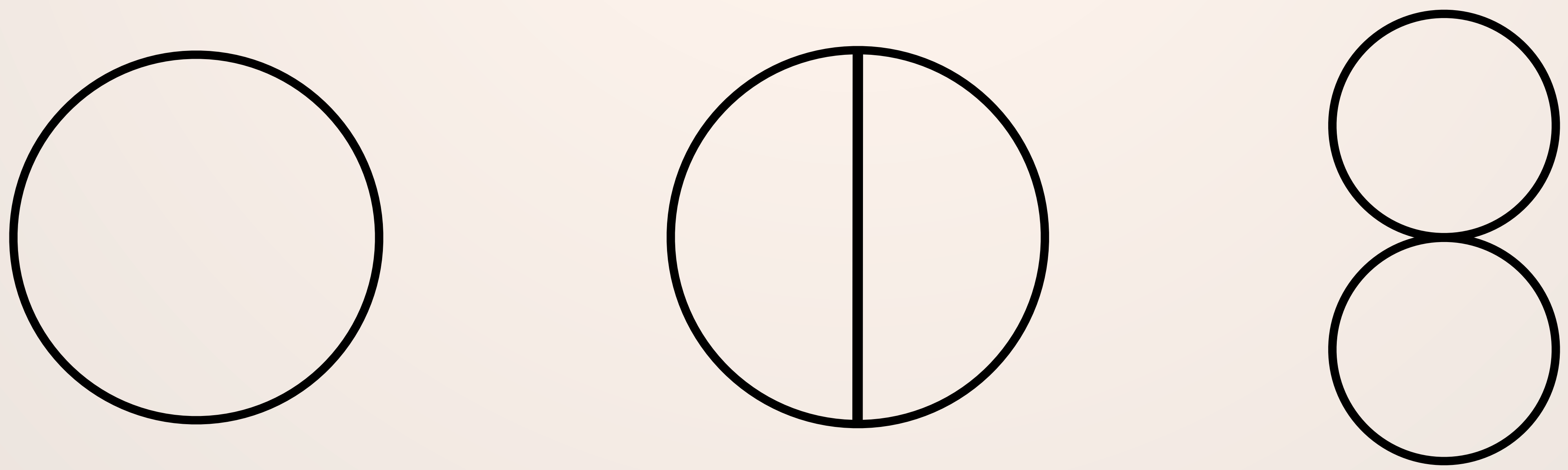
[Baldazzi, Falls, YK, Knorr, 2023]

A Perturbative Approach to Quantum Gravity

- Start with action

$$S = \int \mathrm{d}^4x \sqrt{-g} \left[\frac{R}{16\pi G_0} - \frac{\Lambda_0}{8\pi G_0} + \lambda_E^{(0)} E + \lambda_{C^3}^{(0)} C^3 + \ldots \right]$$

- Calculate observables as perturbative series in Newton coupling
- Compute Feynman diagrams to calculate observables



- But can we see a fixed point for the Einstein-Hilbert terms?

Running Couplings in Quantum Gravity

- β-functions in 4d one-loop quantum gravity in dimensional regularization with MS
  - No power-law divergences

$$\mu \frac{\mathrm{d}\lambda}{\mathrm{d}\mu} = -\lambda\left(2 + A_3 g \lambda\right) + \mathcal{O}\left(g^2\right)$$

$$\mu \frac{\mathrm{d}g}{\mathrm{d}\mu} = g\left(2 - B_2 g \lambda\right) + \mathcal{O}\left(g^3\right)$$

  - $A_3 \neq -B_2$
  - No one-loop fixed point

Running Couplings in Quantum Gravity

- β-functions in 4d one-loop quantum gravity

quart div

$$\mu \frac{\mathrm{d}\lambda}{\mathrm{d}\mu} = -2\lambda - g\left(A_1 + A_2\lambda + A_3\lambda^2\right) + \mathcal{O}\left(g^2\right)$$

quad div

log div

$$\mu \frac{\mathrm{d}g}{\mathrm{d}\mu} = 2g - g^2\left(B_1 + B_2\lambda\right) + \mathcal{O}\left(g^3\right)$$

Various Ways to Regularize QFT

- Momentum Cutoff

$$\int \mathrm{d}^4 p \, \frac{1}{p^2 + m^2} \to \int_0^k \mathrm{d}p \, p^3 \int \mathrm{d}\Omega_4 \, \frac{1}{p^2 + m^2}$$

- Dimensional Regularization

$$\int \mathrm{d}^4 p \frac{1}{p^2 + m^2} \to \int_0^\infty \mathrm{d}p \, p^{d-1} \int \mathrm{d}\Omega_d \frac{1}{p^2 + m^2}$$

- Schwinger proper time cutoff (operator regularization)

$$\int \mathrm{d}^4 p \, \frac{1}{p^2 + m^2} \to \int_{1/k^2}^\infty \mathrm{d}s \int \mathrm{d}^4 p \, e^{-s(p^2+m^2)}$$

Running Couplings in Quantum Gravity

- β-functions in 4d one-loop quantum gravity in dimensional regularization with MS
  - **No power-law divergences**

Is dimensional regularization blind to power-law divergences?

$$\mu\frac{\mathrm{d}\lambda}{\mathrm{d}\mu} = -\lambda\left(2 + A_3 g\lambda\right) + \mathcal{O}\left(g^2\right)$$

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Minimal Subtraction with a Cutoff

- Let us consider a cutoff regulator k

- General form of one-loop renormalization for Newton coupling

$$G_0 = G + G^2 \left[B_1 k^2 + B_2 \Lambda \log \left(\frac{k}{\mu} \right) \right] + \mathcal{O}\left(G^3\right)$$

- We get running coupling from sliding scale dependence

$$\frac{\mathrm{d}}{\mathrm{d}\log\mu} G_0 = 0 \qquad \Rightarrow \qquad \mu \frac{\mathrm{d}g}{\mathrm{d}\mu} = 2g + B_2 g^2 \lambda^2$$

Quadratic divergence?

Non-Minimal Subtraction with a Cutoff

- Sensitivity to power-law divergences can be regained following a non-minimal subtraction, e.g. Niedermaier $G_0(\mu = k) = G$ [Niedermaier, 2009]
- This leads to

$$G_0 = G + G^2 \left[B_1 \left(k^2 - \mu^2 \right) + B_2 \Lambda \log \left(\frac{k}{\mu} \right) \right]$$

with

$$\mu \frac{\mathrm{d}g}{\mathrm{d}\mu} = 2g + 2B_1 g^2 + B_2 g^2 \lambda^2$$

- **We require a non-minimal subtraction to be sensitive to power-law divergences**
    - *Can we use non-minimal renormalization schemes in dimensional regularization?*

Power-Law Sensitive Dimensional Regularization

- Ansatz for non-minimal dimensional regularization

$$G_0 = G + G^2 \left[B_1 \mu^{d-2} + B_2 \Lambda \frac{\mu^{d-4}}{d-4} \right]$$

  - **We require a *non-minimal* scheme that fixes** B_1

Power-Law Sensitive Dimensional Regularization

- How to identify power-law divergences in dimensional regularization?

$$\int \frac{\mathrm{d}^d p}{\pi^{d/2}} \frac{1}{p^2 + m^2} = \int \mathrm{d}p\, p^{d-1} \int \mathrm{d}\Omega_d \frac{1}{p^2 + m^2} = m^{d-2}\Gamma\left(1 - \frac{d}{2}\right)$$

- Integrand on the left-hand side is quadratically divergent in $d = 4$
  - It also contains a logarithmic divergence (linear in m^2)

- The right-hand side is divergent in $d = 2,4,6,\ldots$
  - **Power-law divergences lead to divergences in $d < 4$** [Veltman, 1981]

Power Divergence Subtraction:

Renormalize bare couplings to remove divergences in $d \leq 4$

Power Divergence Subtraction

- Similar scheme first mentioned by Weinberg [Weinberg, 1979]
- First applications to renormalizable field theories [Jack and Jones, 1990] [Al-Sarhi and Jack and Jones, 1990] [Al-Sarhi and Jack and Jones, 1992]
- Rediscovered in nucleon effective field theories [Kaplan et al, 1998]
  - PDS effectively extends valid range of momentum expansion compared to MS
    - Similar behaviour as cutoff regularizations

Power Divergence Subtraction

- Divergences in $d < 4$ correspond to additional finite renormalizations in $d = 4$
    - Valid non-minimal subtraction scheme

- Can be generalized to higher loop orders
    - Power-law divergences of degree N are seen in

$$d_{\text{crit}} = 4 - \frac{N}{L}$$

at L-loop

PDS for Newton's coupling

- General Ansatz for Newton's coupling

$$G_0 = G + G^2 \left[B_1 \frac{\mu^{d-2}}{d-2} + B_2 \Lambda \frac{\mu^{d-4}}{d-4} \right]$$

- This leads to β-functions

$$\mu \frac{\mathrm{d}g}{\mathrm{d}\mu} = (d-2)g - B_1 g^2 - B_2 g^2 \lambda$$

PDS in Einstein-Hilbert

- Renormalization of couplings

$$G_0 = G + G^2 \left[B_1 \frac{\mu^{d-2}}{d-2} + B_2 \Lambda \frac{\mu^{d-4}}{d-4} \right]$$
$$\Lambda_0 = \Lambda + G \left[A_1 \frac{\mu^d}{d} + A_2 \Lambda \frac{\mu^{d-2}}{d-2} + A_3 \Lambda^2 \frac{\mu^{d-4}}{d-4} \right]$$

- β-functions

$$\mu \frac{\mathrm{d}\lambda}{\mathrm{d}\mu} = -2\lambda - A_1 g - A_2 g \lambda - A_3 g \lambda^2$$
$$\mu \frac{\mathrm{d}g}{\mathrm{d}\mu} = (d-2)g - B_1 g^2 - B_2 g^2 \lambda$$

One-Loop Quantum Gravity

- Starting from

$$S = \int \mathrm{d}^4x \sqrt{-g} \left[\frac{R}{16\pi G_0} - \frac{\Lambda_0}{8\pi G_0} \right]$$

we compute one-loop divergences

$$\Gamma^{(1)} = \frac{i}{2} \operatorname{Tr} \log\{\mathscr{H}\} - i \operatorname{Tr} \log\{S^{(2)}_{\mathrm{gh}}\}$$

in $d = 4,2,0$

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in $d = 4, {\color{red}2}, 0$

Quantum Gravity in $d = 2 + \varepsilon$

- First indications for asymptotic safety from perturbation theory in $d = 2 + \varepsilon$

[Kawai and Ninomiya, 1990]
[Jack and Jones, 1991]
[Kawai and Kitazawa and Ninomiya, 1993]
⋮

$$\mu \frac{\mathrm{d}g}{\mathrm{d}\mu} = \varepsilon g - B_1 g^2 + \mathcal{O}\left(g^3\right)$$

  - Extrapolate to $\varepsilon \to 2$
    - UV fixed point!

Quantum Gravity in $d = 2 + \varepsilon$

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  - What about higher loop orders?
    - Problem: Propagator has $\frac{1}{d-2}$ - pole [Kawai and Ninomiya, 1990]

$$\mathcal{K}^{\mu\nu}{}_{\rho\sigma} = -i\left(g^{\mu}_{(\alpha}g^{\nu}_{\beta)} - \frac{1}{2}g^{\mu\nu}g_{\alpha\beta}\right)\left[p^2 - 2\Lambda_0\right]$$

$$\rightarrow \mathcal{P}^{\mu\nu}{}_{\rho\sigma} = \left(g^{\mu}{}_{(\rho}g^{\nu}{}_{\sigma)} - \frac{1}{d-2}g^{\mu\nu}g_{\rho\sigma}\right)\frac{-i}{p^2 - 2\Lambda_0}$$

- This leads to pathologies:
  - Possible misidentification of UV divergences
  - Non-local divergences at two-loop [Jack and Jones, 1991]

Quantum Gravity in $d = 2 + \varepsilon$

- Modify the Hessian by including an artificial mass m:

$$\mathscr{H}^{\mu\nu}{}_{\rho\sigma} = -i\left(g^{\mu}{}_{(\alpha}g^{\nu}{}_{\beta)} - \frac{1}{2}g^{\mu\nu}g_{\alpha\beta}\right)\left[p^2 - 2\Lambda_0\right] + i g^{\mu}{}_{(\alpha}g^{\nu}{}_{\beta)}m^2$$

$$\to \mathscr{P}^{\mu\nu}{}_{\rho\sigma} = g^{\mu}{}_{(\rho}g^{\nu}{}_{\sigma)}\frac{-i}{p^2 - 2\Lambda_0 - m^2} + i g^{\mu\nu}g_{\rho\sigma}\frac{p^2 - 2\Lambda_0}{m^2\left(p^2 - m^2 - 2\Lambda_0\right)}$$

- **Singularity of propagator in $d = 2$ now regulated by m!**

One-Loop Divergences

- Logarithmic divergences, i.e. $d = 4$

$$\Gamma^{(1)}\Big|_{d=4-2\epsilon} = \frac{1}{(4\pi)^2\epsilon}\left[-10\Lambda_0^2 + \frac{13}{3}\Lambda_0 R - \frac{53}{90}E - \frac{7}{20}R_{\mu\nu}R^{\mu\nu} - \frac{1}{120}R^2\right] + \mathcal{O}(1)\,.$$

  - These are universal

One-Loop Divergences

- Quadratic divergences, i.e. $d = 2$

$$\Gamma^{(1)}\Big|_{d=2-2\epsilon} = \frac{1}{(4\pi)\epsilon}\left[-2\Lambda_0 + \frac{13}{6}R\right] + \mathcal{O}(1)\,.$$

- Naive result (without mass regulator):

$$\Gamma^{(1)}\Big|_{d=2-2\epsilon} = \frac{1}{(4\pi)\epsilon}\left[-3\Lambda_0 + \frac{19}{12}R\right] + \mathcal{O}(1)\,.$$

One-Loop Divergences

- Quartic divergences, i.e. $d = 0$

$$\Gamma^{(1)}\Big|_{d=-2\epsilon} = \mathcal{O}(1)\,.$$

  - No quartic divergence present
    - Trace of heat kernel cancels pole in $d = 0$
    - Consequence of using regulator preserving diffeomorphism invariance? [Akhmedov, 2002] [Ossola and Sirlin, 2003]

Renormalization

- One-loop renormalization leads to β-functions

$$\beta_\lambda = \mu \frac{\mathrm{d}\lambda}{\mathrm{d}\mu} = -2\lambda - \frac{28}{3} g\lambda + \frac{4}{3\pi} g\lambda^2$$

$$\beta_g = \mu \frac{\mathrm{d}g}{\mathrm{d}\mu} = 2g - \frac{2 \cdot 26}{3} g^2 - \frac{26}{3\pi} g^2 \lambda$$

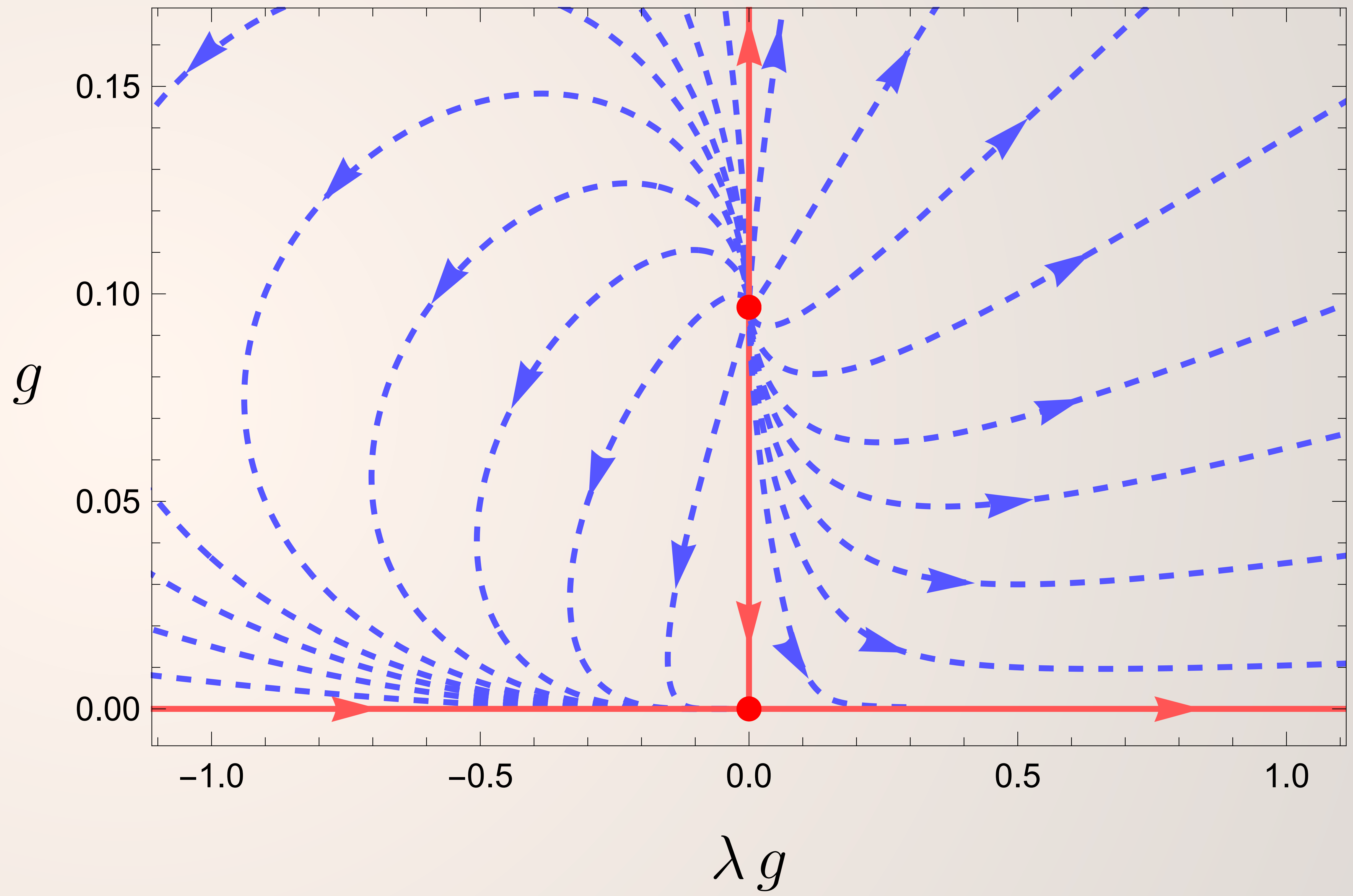
- β-function in $d = 2$ with c scalar fields:

$$\beta_g = -(26 - c)\frac{2}{3} g^2$$ [Falls, 2015]

Phase Diagram of One-Loop Gravity

	λ	g	θ_0	θ_1
$\mathbf{FP}_{\text{Gauss}}$	0	0	2	-2
$\mathbf{FP}_{\text{UV}}$	0	$\frac{3}{26}$	3.077	2
$\mathbf{FP}_{\text{unphys}}$	$-\frac{40\pi}{11}$	$-\frac{11}{78}$	$0.658 + 2.662i$	$0.658 - 2.662i$

- One-loop fixed point exists
  - Critical exponents similar to FRG
  - $\lambda = 0$ is a fixed point of the RG flow



Some Remarks

- Fixed point is generated by quadratic divergences
    - Divergences in $d = 2$ are determining its properties
    - Is the UV of gravity two-dimensional?
- Quartic divergences are absent in this scheme
    - $\lambda = 0$ is a fixed point

Conclusions…

- Study of Asymptotic Safety with power-law sensitive renormalization schemes
  - Dimensional regularization expresses power-law divergences as divergences in lower dimensions
- This provides a one-loop fixed point from dimensional regularization of 4d quantum gravity

...Outlook

- PDS allows for applications to higher orders
    - Fixed points in two-loop quantum gravity?
- What happens if matter is included?
- PDS in other non-renormalizable field theories?

Any Questions…?